Mixed graphs with smallest eigenvalue greater than $-\sqrt{\frac{5+1}{2}}$

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\section*{A B S T R A C T}

The classical problem of characterizing the graphs whose eigenvalues lie in a given interval may date back to the work of J.H. Smith in 1970. Especially, the research on graphs with smallest eigenvalues not less than $-2$ has attracted widespread attention. Mixed graphs are natural generalizations of undirected graphs. In this paper, we completely characterize the mixed graphs with smallest Hermitian eigenvalue greater than $-\sqrt{\frac{5+1}{2}}$. In fact, we found three infinite classes of mixed graphs and 30 scattered mixed graphs.

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\section{1. Introduction}

It is a classical problem in Spectral Graph Theory to characterize the graphs whose eigenvalues lie in a given interval. The research of such problems may date back to the work of Smith in 1970 [18]. This work stimulated the interest of researchers. There are a lot of results in the literature concerning the topic. In 1972, Hoffman [11] obtained all limit points of the spectral radius of nonnegative symmetric matrices smaller than $\sqrt{\frac{5+1}{2}}$. In 1982, Cvetković et al. [5] characterized the graphs whose spectral radius does not exceed $\sqrt{2}+\sqrt{3}$, in 1989, Brouwer and Neumaier [4] determined the graphs with spectral radius between 2 and $\sqrt{2}+\sqrt{3}$ and later, Woo and Neumaier [27] described the structure of graphs whose spectral radii are bounded above by $3\sqrt{2}/2$. For the (signless) Laplacian matrices, Wang et al. [21,22] characterized the graphs whose spectral radii do not exceed 4.5. With respect to the smallest eigenvalues, Hoffman [12] investigated the graphs whose smallest eigenvalue exceeds $-1-\sqrt{2}$, and this work was continued by Taniguchi et al. [19,20,14]. Furthermore, Munemasa et al. [16] showed that all fat Hoffman graphs with smallest eigenvalue at least $-\sqrt{\frac{5+1}{2}}$ (which is just $-1-\tau$ where $\tau$ is the golden ratio) can be described by a finite set of fat $(1-\tau)$-irreducible Hoffman graphs. Especially, the graphs with smallest eigenvalue $-2$ attracted a lot of attention, and we refer the reader to the survey [7] and the book [6]. Recently, Abdollahi et al. [1] classified all distance-regular Cayley graphs with least eigenvalue $-2$ and diameter at most three, and Koolen et al. [13] proved that a connected graph with smallest eigenvalue at least $-3$ and large enough minimal degree is 2- integrable. For the anti-adjacency matrices, Wang et al. [23,24] determined the graphs whose smallest eigenvalues are at least $-2\sqrt{2}$. In this paper, we consider the smallest Hermitian eigenvalue of a mixed graph.

A mixed graph is defined to be an ordered triple $(V,E,A)$, where $V$ is the vertex set, $E$ is the undirected edge set and $A$ is the directed edge set. Note that, if both $uv$ and $vu$ are directed edges, then we regard $(u,v)$ as an undirected edge. Thus, if $(u,v) \in A$ then $(v,u) \notin A$. Clearly, if $A = \emptyset$ then the mixed graph turns to be a graph and if $E = \emptyset$ then the mixed graph

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turns to be an oriented graph. For convenience, we write \( u \leftrightarrow v \) if \( \{u, v\} \in E \) and \( u \rightarrow v \) is \( \{u, v\} \in A \). Let \( M = (V, E, A) \) be a mixed graph with \( V = \{v_1, v_2, \ldots, v_n\} \). The underlying graph \( \Gamma(M) \) is a graph with vertex set \( V \) and two vertices \( u \sim v \) if \( u \leftrightarrow v \) or \( u \rightarrow v \) or \( v \rightarrow u \). For \( U \subseteq V \) and \( W \subseteq V \setminus U \), denote by \( N_W(U) = \{w \mid w \in W, u \sim w \text{ in } \Gamma(M) \text{ for some } u \in U\} \). Especially, if \( U = \{u\} \) then \( N_W(u) \) is the set of neighbors of \( u \) in \( W \). Moreover, denote by \( N^+_W(u) = \{w \mid u \rightarrow w\} \) and \( N^-_W(u) = \{w \mid u \leftarrow w\} \). It is clear that \( N_W(u) = N^+_W(u) \cup N^-_W(u) \cup N^0_W(u) \). As usual, we always write \( P_n, C_n, K_{n_1,n_2,\ldots,n_k} \) to denote the path, the cycle, and the complete multipartite graph of the corresponding orders. For two graphs \( G \) and \( H \), the union \( G \cup H \) is the graph with vertex set \( V(G) \cup V(H) \) and edge set \( E(G) \cup E(H) \). The join \( G*H \) is the graph obtained from \( G \cup H \) by adding all edges between \( G \) and \( H \). The distance \( d_G(u,v) \) of two vertices \( u, v \in V(G) \) is the length of a shortest path from \( u \) to \( v \). The diameter of \( G \) is the largest distance in \( G \), denoted by \( d(G) \). All other notations not mentioned here are standard and can be found in [8].

We always write \( M_G \) for \( M \) when the underlying graph \( \Gamma(M) = G \). Moreover, for a graph \( G \), denote by \( M_G \) the set of mixed graphs with underlying graph \( G \). If \( M_G = G \) then we write \( G \) for \( M_G \). The mixed graph \( M_G \) is connected if \( G \) is connected and we always consider connected mixed graphs in this paper. The diameter of \( M_G \) is defined to be the diameter of \( G \), denoted by \( d(M_G) \). For a subset \( U \subseteq V \), the mixed subgraph induced by \( U \) is the mixed graph \( M_G[U] = \langle U, E', A' \rangle \) with \( E' = \{(u, v) \mid u, v \in U, \{u, v\} \in E\} \) and \( A' = \{(u, v) \mid u, v \in U, \{u, v\} \in A\} \). As usual for a vertex \( v \), the (mixed) graph \( G - v \) (resp. \( M_G - v \)) is the induced (resp. mixed) subgraph obtained from \( G \) (resp. \( M_G \)) by deleting the vertex \( v \) and associated edges. The Hermitian matrix of \( M_G \) is defined to be a square matrix \( H(M_G) = [h_{st}]_{n \times n} \) with

\[
h_{st} = \begin{cases} 1, & v_s \leftrightarrow v_t, \\ i, & v_s \rightarrow v_t, \\ -i, & v_t \rightarrow v_s, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( i = \sqrt{-1} \). This matrix was proposed by Liu and Li [15] and Guo and Mohar [9] independently. Since \( H(M_G) \) is a Hermitian matrix, all eigenvalues of \( H(M_G) \) are real and listed as \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{\min} \). The collection of such eigenvalues is the spectrum of \( H(M_G) \). The Hermitian spectrum of the mixed graph \( M_G \) is just the spectrum of \( H(M_G) \), denoted by \( \text{Sp}(M_G) \). Two mixed graphs \( M_G, M'_G \in M_G \) are switching equivalent if there exists a diagonal matrix \( D \) whose entries belong to \( \{\pm 1, \pm i\} \) such that \( H(M'_G) = DH(M_G)D^* \). It is clear that switching equivalence is an equivalence relation. Thus, denote by \( [M_G] \) the equivalence class containing \( M_G \) with respect to switching equivalence. Obviously, all graphs in \( [M_G] \) share the same spectrum. Recently, Wissing and Dam [26] determined all mixed graphs with exactly one negative eigenvalue. Guo and Mohar [10] determined all mixed graphs with \( \lambda_1 < 2 \) and Yuan et al. [28] characterized all mixed graphs with \( \lambda_1 \leq 2 \) when \( G \) contains no cycles of length 4.

In this paper, we completely determine the connected mixed graphs with smallest Hermitian eigenvalue greater than \(-\sqrt{\frac{3}{2}}\). In fact, we found three infinite classes of mixed graphs and 30 scattered mixed graphs (see Theorem 5).

2. Preliminaries

We first present the famous interlacing theorem with respect to Hermitian matrices.

**Lemma 1** ([3]). Let the matrix \( S \) of size \( n \times m \) be such that \( S^tS = I_m \) and let \( H \) be a Hermitian matrix of size \( n \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = \lambda_{\min} \). Set \( B = S^tHS \) and let \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \) be the eigenvalues of \( B \). Then the eigenvalues \( \mu_i \) interlace the eigenvalues \( \lambda_j \), that is, \( \lambda_i \geq \mu_i \geq \lambda_{n-m+i} \) for \( i = 1, 2, \ldots, m \).

The following result is immediate from Lemma 1.

**Corollary 1.** Let \( M_G \) be a mixed graph with underlying graph \( G \). If \( M_H \) is a mixed induced subgraph of \( M_G \), then the eigenvalues of \( M_H \) interlace those of \( M_G \).

Next, we introduce another powerful tool in spectral graph theory: the equitable partition. Let \( M_G \) be a mixed graph on \( n \) vertices with underlying graph \( G \). Let \( \pi: V(G) = V_1 \cup V_2 \cup \cdots \cup V_s \) be a partition of \( V(G) \) with \( |V_i| = n_i \) and \( n = n_1 + n_2 + \cdots + n_s \). For \( 1 \leq i, j \leq s \), denote by \( H_{ij} \) the submatrix of \( H(M_G) \) whose rows correspond to \( V_i \) and columns correspond to \( V_j \). Therefore, the Hermitian matrix \( H(M_G) \) can be written as \( H(M_G) = [H_{ij}] \). Denote by \( b_{ij} = \frac{1}{n_i} H_{ij}1/n_i \) the average row-sums of \( H_{ij} \), where \( 1 \) denotes the all-one vector. The matrix \( H_{\pi} = (b_{ij})_{s \times s} \) is called the quotient matrix of \( H(M_G) \). If, for any \( i, j \), the row-sum of \( H_{ij} \) corresponding to any vertex \( v \in V_i \) equals \( b_{ij} \), then \( \pi \) is called an equitable partition of \( M_G \). Let \( \delta_{V} \) be a vertex indexed by \( V(G) \) such that \( \delta_{V_i}(v) = 1 \) if \( v \in V_i \) and \( 0 \) otherwise. The matrix \( P = [\delta_{V_1}, \delta_{V_2}, \ldots, \delta_{V_s}] \) is called the characteristic matrix of \( \pi \). If \( \pi \) is an equitable partition, then \( H(M_G)P = PH_{\pi} \). It leads to the following famous result.

**Lemma 2** ([8, Theorem 9.3.3, page 197]). Let \( M_G \) be a mixed graph and \( \pi \) an equitable partition of \( M_G \) with quotient matrix \( H_{\pi} \) and characteristic matrix \( P \). Then the eigenvalues of \( H_{\pi} \) are also eigenvalues of \( H(M_G) \). Furthermore, \( H(M_G) \) has the following two kinds of eigenvectors:
(i) the eigenvectors in the column space of $P$, and the corresponding eigenvalues coincide with the eigenvalues of $H_\pi$;
(ii) the eigenvectors orthogonal to the columns of $P$, i.e., those eigenvectors sum to zero on each cell of $\pi$.

Let $\mathcal{H}$ be a set of graphs. A graph $G$ is called $\mathcal{H}$-free if no induced subgraphs of $G$ are in $\mathcal{H}$. If $\mathcal{H} = \{H\}$ then an $\mathcal{H}$-free graph $G$ is also called an $H$-free graph. Recall that a $P_4$-free graph is called a cograph. The following result reveals the structure of cographs.

**Lemma 3** ([17]). If $G$ is a connected $P_4$-free graph, then $G$ is the join of two graphs, that is, $G = G_1 \lor G_2$ for some graphs $G_1$ and $G_2$ with $|V(G_1)|, |V(G_2)| \geq 1$.

We determine two types of $\mathcal{H}$-free graphs when $\mathcal{H}$ contains some simple graphs.

**Lemma 4.** If $G$ is a $\{P_3, 3K_1, K_2 \cup K_1\}$-free graph then $G \in \{2K_1, Kn \mid n \geq 1\}$; if $G$ is a $\{P_3, 3K_1, K_3\}$-free graph then $G \in \{K_1, K_2, 2K_1, 2K_2, K_1, K_2 \cup K_2\}$.

**Proof.** It is clear that, if a graph $G$ is $P_3, 3K_1$-free, then it is the union of at most two complete graphs. Thus, we have $G \in \{2K_1, Kn \mid n \geq 1\}$ if $G$ is additional $K_2 \cup K_1$-free, and $G \in \{K_1, K_2, 2K_1, 2K_2, K_1, K_2 \cup K_2\}$ if $G$ is additional $K_3$-free. □

Guo and Mohar introduced the so-called four-way switching to generate switching equivalent graphs [9]. A **four-way switching** is the operation of changing a mixed graph $M_G$ into the mixed graph $M_{G'}$ by choosing an appropriate diagonal matrix $S$ with $S_{ij} \in \{\pm 1, \pm i\}$ and setting $H(M_{G'}) = S^{-1}H(M_G)S$. Let $G$ be a graph and $X$ an edge cut such that $G - X = G_1 \cup G_2$ and $V_1 = V(G_1)$ and $V_2 = V(G_2)$. For a mixed graph $M_G = (V, E, A)$, define $X^+ = \{(v_1, v_2) \mid v_1, v_2 \in V \}$ and $X^- = \{(v_2, v_1) \mid v_1, v_2 \in V \}$. The cut $X$ is called a coincident cut of the mixed graph $M_G$ if $X^+ \subseteq A$ or $X^- \subseteq A$ or $X \subseteq E$. If $X$ is a coincident cut of $M_G$, then the $X$-switching of $M_G$ is the mixed graph $M_G[X] = (V, E', A')$ with $E' = E \setminus X \cup X$ and $A' = A \setminus (X^+ \cup X^-)$. Note that $M_G[X] = M_{G'}$ if $X \subseteq E$. From four-way switching, the following results are obtained.

**Lemma 5** ([9]). Let $M_G$ be a mixed graph. If $X$ is a coincident cut of $M_G$, then $M_G$ and $M_G[X]$ are switching equivalent and thus $Sp(M_G) = Sp(M_G[X])$.

If $G$ is a forest, then each edge is a cut. Moreover, each edge is a coincident cut of any mixed graph $M_G$. Thus, Lemma 5 implies the following result.

**Corollary 2** ([9]). If $G$ is a forest, then $Sp(M_G) = Sp(G)$ for any mixed graph $M_G \in M_G$.

Note that mixed graphs could be viewed as the so-called gain graphs. Let $T_4 = \{\pm 1, \pm i\}$ be the group of the fourth roots of unity. For an undirected graph $G$ with vertex set $V$ and edge set $E$, the $T_4$-gain graph is a triple $\Phi = (G, T_4, \varphi)$ consisting the underlying graph $G$, the gain group $T_4$ and a map $\varphi: E \rightarrow T_4$ such that $\varphi((u, v)) = \varphi((v, u))^{-1}$ called the gain function. The mixed graphs are just the $T_4$-gain graphs with $\varphi(\tilde{E}) \in \{1, \pm i\}$. For a mixed graph $M_G$ with Hermitian matrix $H = [h_{uv}]_{u,v \in X}$, let $C = v_1 v_2 \cdots v_n$ be a cycle in $G$, denote by $h_{M_G}(C) = h_{v_1v_2}h_{v_2v_3} \cdots h_{v_nv_1}$. Moreover, if $h_{M_G}(C) = 1$ then we say $C$ is positive. In [2], the authors investigated $T_4$-gain graphs, and from Propositions 1 and 2 in [2], we get the following useful result, which is also obtained by Wang and Yuan [25].

**Lemma 6** ([2,25]). Let $M_G$ and $M_G'$ be two mixed graphs sharing the same underlying graph $G$. If every induced cycle $C$ in $G$ is positive, then $M_G \in \{G\}$.

### 3. Mixed graphs with $\lambda_{\text{min}} > -\sqrt{2} + 1$

In this part, we first investigate the mixed triangles in mixed graphs whose underlying graph is a complete graph. Next, we get all mixed graphs with smallest eigenvalue not less than $-\sqrt{2}$. At last, we completely determine the mixed graphs with smallest eigenvalue greater than $-\sqrt{2} + 1 \approx -1.618$.

It is easy to verify that there are seven types of mixed triangles and fourteen types of mixed quadrangles, and we present them in Fig. 1 together with their smallest eigenvalues. The following results are immediate from Lemma 1 and Fig. 1.

**Lemma 7.** Let $M_G$ be a mixed graph with smallest eigenvalue $\lambda_{\text{min}}$. If $\lambda_{\text{min}} > -\sqrt{3}$, then any mixed triangle in $M_G$ belongs to $\{K_3, K_2^2, K_2^3\}$.

**Lemma 8.** Let $M_G$ be a mixed graph with smallest eigenvalue $\lambda_{\text{min}}$. If $\lambda_{\text{min}} \geq -1.84$, then any induced mixed quadrangle in $M_G$ belongs to $\{C_4^1, C_4^2, C_4^3\}$.
In what follows, we always denote $C_3 = \{K_3, K_{3,2}^2, K_{3,3}^2\}$ and $C_4 = \{C_4^1, C_4^2, C_4^3\}$. The mixed triangles $K_3, K_{3,2}^2$ and $K_{3,3}^2$ play an important role in determining the orientations of a mixed graph, especially when all induced cycles (if exist) of the underlying graph are triangles. Recall that a chordal graph is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. The following result characterizes a class of mixed chordal graphs switching equivalent to their underlying graphs.

**Theorem 1.** Let $G$ be a chordal graph. If $M_G$ is a mixed graph in which each mixed triangle belongs to $C_3$, then $M_G \in \langle G \rangle$, i.e., $M_G$ is switching equivalent to $G$.

**Proof.** Without loss of generality, assume that $G$ is connected. According to Lemma 5, it suffices to show that $M_G$ has a coincident cut $X$ such that $V(G - X) = U \cup W$ satisfying that all edges in $U$ and $W$ are undirected.

We prove the statement by induction on $n = |V(G)|$. The statement holds for $n = 3$ clearly. Assume that the statement holds for $n - 1$ with $n \geq 4$ and we prove it holds for $n$. It is well-known that a chordal graph has a perfect elimination ordering, which is an ordering of the vertices such that, for each vertex $v$, the vertex $v$ and the neighbors of $v$ that occur after $v$ in the order form a clique. Assume that $\{v_1, v_2, \ldots, v_n\}$ is a perfect ordering. By the inductive hypothesis, $M_G - v_1$ has a coincident cut $X$, say $V(G - v_1) = U \cup W$ such that the edges in $U$ and $W$ are undirected. According to the definition of $X$, we have either all edges between $U$ and $W$ are undirected or they have the same direction. Therefore, we divide two cases to discuss.

**Case 1.** All edges between $U$ and $W$ are undirected.

For any $u \in N_U(v_1)$ and $w \in N_W(v_1)$, since $v_1, u, w$ form a clique and $M_G[v_1, u, w] \in C_3$, we have either $v_1 \leftrightarrow u, w, v_1 \leftrightarrow u, w$ or $u, w \leftrightarrow v_1$. If the first case occurs, then there is nothing to prove. If the second case occurs, then, for any $u' \in N_U(v_1)$, we have $v \rightarrow u'$ since $v_1, u, u'$ form a clique and $M_G[v_1, u, u'] \in C_3$. Similarly, we have $v_1 \rightarrow w'$ for any $w' \in N_W(v_1)$. Therefore, all edges between $v_1$ and $U \cup W$ form the desired coincident cut. If the last one occurs, one can similarly verify that all edges between $v_1$ and $U \cup W$ form the desired coincident cut.

**Case 2.** All edges between $U$ and $W$ have the same direction, say $u \rightarrow v$ for any $u \in U$ and $w \in W$ with $u \sim v$ in $G$.

For any $u \in N_U(v_1)$ and $w \in N_W(v_1)$, since $v_1, u, w$ form a clique and $M_G[v_1, u, w] \in C_3$, we have either $v_1 \leftrightarrow u, w, v_1 \leftrightarrow w, u \leftrightarrow v_1$ and $v_1 \rightarrow v_1$. If the former occurs, then, for any $u' \in N_U(v_1)$, we have $v_1 \rightarrow u'$ since $v_1, u, u'$ form a clique and $M_G[v_1, u, u'] \in C_3$. Similarly, we have $v_1 \rightarrow w'$ for any $w' \in N_W(v_1)$. Therefore, all edges between $v_1 \cup U$ and $W$ form the desired coincident cut. If the latter occurs, one can similarly verify that all edges between $U$ and $v_1 \cup W$ form the desired coincident cut.

The proof is completed. $\square$
Remark 1. We prove Theorem 1 by investigating the structure of chordal graphs, and omit any reference to the spectral theory of complex unit gain graphs. In fact, Theorem 1, as well as Corollary 2, is immediate from Lemma 6 since all induced cycles of a chordal graph are triangles and all of the triangles are in $C_3$ which are positive.

For nonnegative integers $s, t, n$ with $n = s + t$, denote by $K_n[s, t]$ the mixed graph obtained from $K_s \cup K_t$ by adding all arcs from the vertices of $K_s$ to those of $K_t$. We may assume that $K_n[0, 0] = K_n[0, n] = K_n$. From Theorem 1 it immediately follows that $K_n[s, t]$ is switching equivalent to $K_n$. In fact, we will show that $[K_n] = [K_n[s, t]]$ for $s \geq 0, s + t = n$ and give a characterization of the graph set $[K_n]$.

Lemma 9. Let $M_{K_n}$ be a mixed graph on $n \geq 3$ vertices in which any mixed triangle belongs to $C_3$. If $M_{K_n}$ contains $K_3^{2, 2}$, then $M_{K_n} \in \{K_n[s, t] \mid s \geq 2, t \geq 1, s + t = n\}$.

Proof. Assume that $u, v, w \in V(M_{K_n})$ induce a $K_3^{2, 2}$ with $u \leftrightarrow w, v \leftrightarrow w$ and $u \leftrightarrow v$. For any vertex $x \in V(M_{K_n}) \setminus \{u, v, w\}$ (if exists), we have either $x \leftrightarrow w$ or $x \leftrightarrow w$ since otherwise $M_{K_n}[u, w, x] \notin C_3$. By noticing $M_{K_n}[u, x, w], M_{K_n}[v, x, w] \in C_3$, one can easily verify that $v \leftrightarrow x$ and $u \rightarrow x$ if $x \leftrightarrow w$, and $x \leftrightarrow v$ and $u \leftrightarrow x$ if $x \leftrightarrow w$.

Denote by $V_1 = \{x \in V(M_{K_n}) \mid x \leftrightarrow w\}$ and $V_2 = \{x \in V(M_{K_n}) \mid x \leftrightarrow x\}$. It is clear that $u, v \in V_2$ and $V = V_1 \cup V_2$. Let $V_3 = \{x \in V(M_{K_n}) \mid x \leftrightarrow x\}$. For any two vertices $x_1, x_2 \in V_1 \setminus \{w\}$, we have $x_1 \leftrightarrow x_1$, $x_2 \leftrightarrow w$ and $M_{[x_1, x_2, w]} \in C_3$. Similarly, we have $x_2 \leftrightarrow x_2$ for any $x_2, x_2 \in V_2$. Moreover, for any $x_1 \in V_1 \setminus \{w\}$ and $x_2 \in V_2$, we have $x_2 \leftrightarrow x_1$ since $x_1 \leftrightarrow w, x_2 \rightarrow w$ and $M_{[x_1, x_2, w]} \in C_3$. Thus, $M_{K_n} = K_n[s, t]$ where $s = |V_2| \geq 2$ and $t = |V_1| \geq 1$. □

Similarly, we get the following result.

Lemma 10. Let $M_{K_n}$ be a mixed graph on $n \geq 3$ vertices in which any mixed triangular belongs to $C_3$. If $M_{K_n}$ contains $K_3^{2, 3}$, then $M_{K_n} \in \{K_n[s, t] \mid s \geq 1, t \geq 2, s + t = n\}$.

Proof. Assume that $u, v, w \in V(M_{K_n})$ induce a $K_3^{2, 3}$ with $u \leftrightarrow w, v \leftrightarrow w$, and $u \leftrightarrow v$. For any vertex $x \in V(M_{K_n}) \setminus \{u, v, w\}$ (if exists), we have either $x \leftrightarrow w$ or $x \leftrightarrow w$ since otherwise $M_{K_n}[u, w, x] \notin C_3$. Note that $M_{K_n}[u, x, w], M_{K_n}[v, x, w] \in C_3$. We have $x \leftrightarrow u$ and $x \leftrightarrow v$. If $x \leftrightarrow w$, and $x \leftrightarrow u$ and $x \leftrightarrow v$ if $x \leftrightarrow v$. Let $V_3 = \{x \in V(M_{K_n}) \mid x \leftrightarrow x\}$. For any $x \in V(M_{K_n}) \setminus \{w\}$, we have $x \leftrightarrow x$, $x \leftrightarrow w$ and $M_{[x, x, w]} \in C_3$. Clearly, $V(M_{K_n}) = V_3 \cup V_4$, $u, v \in V_3$. Taking $x_3, x_4 \in V_3$ and $x_3, x_4 \in V_4$, we get $x_3 \leftrightarrow x_3$ and $x_4 \leftrightarrow x_4$. Therefore, $V_3$ and $V_4$ induce a clique, respectively, and $|V_3| \geq 1, |V_4| \geq 2$. Moreover, we also have $x_3 \leftrightarrow x_4$ for any $x_3 \in V_3, x_4 \in V_4$. Therefore, we get $M_{K_n} = K_n[s, t]$ with $s = |V_3| \geq 1$ and $t = |V_4| \geq 2$. □

Lemmas 9 and 10 yield the following result.

Theorem 2. Let $M_{K_n}$ be a mixed graph with underlying graph $K_n$ and $n \geq 3$. Then the following statements are equivalent:

(i) any mixed triangle of $M_{K_n}$ belongs to $C_3$;
(ii) $M_{K_n} \in \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$;
(iii) $M_{K_n} \in \{K_n\}$.

Proof. Firstly, assume that any triangle of $M_{K_n}$ belongs to $C_3$. Lemma 9 and Lemma 10 indicate that $M_{K_n} \in \{K_n[s, t] \mid s \geq 1, t \geq 1, s + t = n\}$ when $G_{C}$ contains $K_3^{2, 2}$ or $K_3^{2, 3}$. If $M_{K_n}$ contains neither $K_3^{2, 2}$ nor $K_3^{2, 3}$, then any mixed triangle of $M_{K_n}$ is $K_3$, and thus $M_{K_n} = K_n = K_n[0, n]$. Conversely, one can easily verify that any mixed triangle of $K_n[s, t]$ belongs to $C_3$. Thus, (i) $\Leftrightarrow$ (ii).

Next we will show $[K_n] = \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$. It is clear that $\{K_n[s, t] \mid s, t \geq 0, s + t = n\} \subseteq [K_n]$. It suffices to show that $[K_n] \subseteq \{K_n[s, t] \mid s, t \geq 0, s + t = n\}$. By the arguments above, it only needs to show that any mixed triangle in $M_{K_n}$ belongs to $C_3$ for any $M_{K_n} \in \{K_n\}$. Assume that $H(M_{K_n}) = [h_{ij,k}]$ for a mixed graph $M_{K_n} \in \{K_n\}$. Since $M_{K_n} \in \{K_n\}$, there exists a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_j \in \{\pm 1, \pm 1\}$ such that $D^*H(M_{K_n})D^* = H(K_n)$. Therefore, for any $[u, v, w] \subseteq V(M_{K_n})$, we have

$$
\left(\begin{array}{ccc}
d_u & d_v & d_w \\
0 & h_{uv} & h_{uw} & h_{vw} \\
h_{uw} & 0 & h_{vu} & h_{uw} \end{array}\right)
\cdot
\left(\begin{array}{ccc}
d_u & d_v & d_w \\
h_{uw} & 0 & h_{vw} \\
h_{wu} & h_{uw} & 0 \end{array}\right)
=\left(\begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \end{array}\right).
$$

It leads to $d_u h_{uw} d_w = 1, d_u h_{uv} d_v = 1$ and $d_v h_{vw} d_w = 1$. Thus, we have $h_{uv} h_{vw} h_{wu} = 1$. It implies that either exactly one of $h_{uv}, h_{vw}, h_{wu}$ equals to 1 or all of them equal to 1. If the former case happens, say $h_{uv} = 1$, then $\{h_{vw}, h_{wu}\} = \{\pm 1\}$, which means $M_{K_n}[u, v, w] = K_3^{2, 2}$ or $K_3^{2, 3}$. If the latter case happens, then $M_{K_n}[u, v, w] = K_3$. Therefore, (ii) $\Leftrightarrow$ (iii).

The proof is completed. □

Now we give a simple application of Theorem 2 as follows.
Theorem 3. Let $M_G$ be a connected mixed graph on $n$ vertices. Then $\lambda_{\min}(M_G) > -\sqrt{2}$ if and only if $M_G \in \{K_n[s,t] | s, t \geq 0, s + t = n\}$.

Proof. Theorem 2 implies that the mixed graph $K_n[s,t]$ has the spectrum $[n-1, [-1]^{n-1}]$, and the sufficiency follows. Now we consider the necessity. Assume that $M_G$ is a mixed graph on $n$ vertices with $\lambda_{\min}(M_G) > -\sqrt{2}$. Since $\text{Sp}(P_3) = [\pm \sqrt{2}, 0]$, the path $P_3$ cannot be an induced subgraph of $G$ due to Corollary 1. Thus, we have $G = K_n$. Furthermore, since $\lambda_{\min}(M_G) > -\sqrt{2} > -\sqrt{3}$, Lemma 7 also implies that each triangle in $M_G$ belongs to $C_3$. Thus, we have $M_G \in \{K_n[s,t] | s, t \geq 0, s + t = n\}$ by Theorem 2. \qed

Theorem 3 gives the characterization of mixed graphs with $\lambda_{\min} > -\sqrt{2}$. In what follows, we will further determine the mixed graphs with $\lambda_{\min} \geq -\sqrt{2}$.

Lemma 11. Let $M_G$ be a connected mixed graph on $n$ vertices. If $\lambda_{\min}(M_G) \geq -\sqrt{2}$, then $G$ is $\{P_3 \cup K_1, (K_2 \cup K_1)\}$-free.

Proof. Suppose to the contrary that $G$ contains induced $H$ for $H \in \{P_3 \cup K_1, (K_2 \cup K_1)\}$. Therefore, Corollary 1 means that $\lambda_4(M_H) \geq -\sqrt{2} > -\sqrt{3}$, and thus each mixed triangle of $M_H$ belongs to $C_3$. Note that $H$ has no cycle with length greater than 3. Theorem 1 implies that $\lambda_4(M_H) = \lambda_4(H)$, which equals to $\lambda_4(P_3 \cup K_1) = -1.56 < -\sqrt{2}$ or $\lambda_4((K_2 \cup K_1) \cup K_1) = -1.48 < -\sqrt{2}$, a contradiction. \qed

By Lemma 11, we get the following result.

Theorem 4. Let $M_G$ be a connected mixed graph on $n \geq 4$ vertices. Then $\lambda_{\min}(M_G) \geq -\sqrt{2}$ if and only if $M_{Kn} \in \{K_n[s,t] | s, t \geq 0, s + t = n\} \cup C_4$.

Proof. The sufficiency is immediate and we show the necessity in what follows. We divide two cases to discuss.

Case 1. $G$ is $P_3$-free.

In this case, we have $G = K_n$. Since $\lambda_{\min}(M_G) \geq -\sqrt{2} > -\sqrt{3}$, any mixed triangle in $M_G$ belongs to $C_3$ by Lemma 7. Thus, Theorem 2 means $M_G \in \{K_n[s,t] | s, t \geq 0, s + t = n\}$.

Case 2. $G$ is not $P_3$-free.

In this case, suppose that there exists $u, v, w \in V(G)$ such that $G[u, v, w] = P_3$ with $u \sim v$ and $v \sim w$. Note that $\lambda_4(P_3) \approx -1.618 < -\sqrt{2}$ and $\lambda_4(K_{1,3}) = -\sqrt{3} < -\sqrt{2}$. Corollary 1 implies that $G$ is $[P_4, K_{1,3}]$-free, and thus the diameter $d(G) = 2$. Therefore, each vertex $y \in V(G) \setminus \{u, v, w\}$ of $V(G)$ is adjacent to at least one vertex of $\{u, v, w\}$. If $y$ is adjacent to exactly one vertex of $\{u, v, w\}$, then $G$ either contains an induced $P_4$ or $K_{1,3}$, which is impossible. If $y$ is adjacent to all the vertices $\{u, v, w\}$, then $G[u, v, w, y] = P_3 \cup K_1$, which contradicts Lemma 11. Thus, $y$ is adjacent to exactly two vertices of $\{u, v, w\}$. If $y \sim u, v$ or $y \sim v, w$, then $G[u, v, w, y] = (K_2 \cup K_1) \cup K_1$, which contradicts Lemma 11. Thus, $y \sim u, w$, that is $G[u, v, w, y] = C_4$. Next, we claim that $n = 4$. Otherwise, there exists another vertex $y' \in V(G) \setminus \{u, v, w, y\}$. By regarding $y'$ as $y$, we have $G[u, v, w, y'] = C_4$. Therefore, we have $G[u, v, w, y'] = K_{1,3}$ when $y \sim y'$ and $G[u, v, w, y'] = (K_2 \cup K_1) \cup K_1$ when $y \sim y'$, which are all impossible. Therefore, we have $G = C_4$, and thus $M_G \in C_4$ by Fig. 1.

This completes the proof. \qed

In what follows, we characterize the mixed graph $M_G$ with $\lambda_{\min}(M_G) > -\frac{1+\sqrt{2}}{2}$. We first find some structural constraints on the underlying graph $G$.

Lemma 12. If $M_G$ is a mixed graph with underlying graph $G = K_{m,n}$, then $\lambda_{\min}(M_G) \leq -\frac{1+\sqrt{2}}{2}$ except for $G \in \{K_2, K_{1,2}, K_{2,2}\}$.

Proof. Assume $\lambda_{\min}(M_G) > -\frac{1+\sqrt{2}}{2}$, then $G$ has no $K_{1,3}$ as an induced subgraph since $\lambda_3(K_{1,3}) = -\sqrt{3} < -\frac{1+\sqrt{2}}{2} \approx -1.618$. This leads to $G \in \{K_2, K_{1,2}, K_{2,2}\}$. \qed

By applying Theorem 1, we get the following result.

Lemma 13. Let $M_G$ be a connected mixed graph on $n$ vertices. If $\lambda_{\min}(M_G) > -\frac{1+\sqrt{2}}{2}$ then $G$ is $\{P_4, K_{1,3}, K_{2,3}, 2K_1 \cup K_{1,2}, K_2 \cup 3K_1, K_2 \cup (K_2 \cup K_1), K_2 \cup K_{1,2}, 2K_1 \cup K_{1,2}\}$-free.

Proof. By Corollary 2, we have $\lambda_4(P_4) = \lambda_4(P_4) < -\frac{1+\sqrt{2}}{2}$ for any $M_{P_4} \in M_{P_4}$ and $\lambda_4(M_{K_{1,3}}) = \lambda_4(K_{1,3}) = -1.73 < -\frac{1+\sqrt{2}}{2}$ for any $M_{K_{1,3}} \in M_{K_{1,3}}$. Thus, Corollary 1 implies that $G$ is $\{P_4, K_{1,3}\}$-free. Suppose to the contrary that $G$ contains an induced $K_{2,3}$. Corollary 1 indicates that $\lambda_3(M_{K_{2,3}}) > -\frac{1+\sqrt{2}}{2}$, which contradicts Lemma 12.

Suppose to the contrary that $G$ contains an induced $K_1 \cup K_{2,2}$ labeled as Fig. 2. Since $M = M_{K_1 \cup K_{2,2}}$ has smallest eigenvalue greater than $-\frac{1+\sqrt{2}}{2}$, Lemma 7 implies all mixed triangles of $M$ belong to $C_2$ and Lemma 8 implies all quadrangles of $M$ belong to $C_4$. If the mixed induced quadrangle $M_{K_{2,2}}$ is equal to $C_4$, then we have either $u_1 \rightarrow v$ or $v \rightarrow u_2$ since
Thus, we assume that one of the eigenvalues $\lambda_1$ and $\lambda_2$ is equal to $C_4$ or $C_4^2$. Then, $M \in \{M_1, M_2, M_3\}$ whose smallest eigenvalues are all $-2 < -\frac{\sqrt{5}}{2}$, a contradiction.

Suppose to the contrary that $G$ contains an induced subgraph $H$ in $K_5 \setminus 3K_1$, $K_2 \setminus (K_2 \cup K_1), K_2 \setminus (K_1 \cup K_2), 2K_1 \setminus K_3$. Therefore, $M_G$ contains a mixed induced graph $M_H$ with order 5. Corollary 1 indicates that $\lambda_m(M_H) \geq \lambda_m(M_G) > -\frac{1+\sqrt{5}}{2}$. Thus, each mixed triangle in $M_H$ belongs to $C_3$. Note that $H$ contains no cycle of length greater than 3. Theorem 1 implies that $\lambda_m(M_H) = \lambda_m(H)$. It leads to a contradiction since $\lambda_5(K_5 \setminus 3K_1) = -2, \lambda_5(K_2 \setminus (K_2 \cup K_1)) = -1.68, \lambda_5(K_2 \setminus (K_1 \cup K_2)) = -1.65$ and $\lambda_5(2K_1 \setminus K_3) = -1.65$ which are all smaller than $-\frac{1+\sqrt{5}}{2}$. \hfill \Box

From Lemma 13, we determine the underlying graphs of $M_G$ with smallest eigenvalue greater than $-\frac{1+\sqrt{5}}{2}$.

**Lemma 14.** Let $M_G$ be a connected mixed graph on $n$ vertices. If $\lambda_{\min}(M_G) > -\frac{1+\sqrt{5}}{2}$, then $G$ belongs to

$$\{K_2, K_1 \cup K_2, 2K_2 \setminus K_1, (K_2 \cup K_1) \setminus 2K_1\} \bigcup \{(K_3 \cup K_4) \setminus K_1 \mid s \geq 0, s + t = n - 1\}.$$

**Proof.** We may assume that $n \geq 2$ since there is nothing to prove when $n = 1$. From Lemma 13, we have $G$ is $P_4$-free and thus $G = X \setminus Y$ with $|X|, |Y| \geq 1$ due to Lemma 3. If both $X$ and $Y$ have no edge, then $G = K_{n,n}$ and thus $G \in \{K_2, K_1, K_{2,2}\}$ due to Lemma 12, where both $K_2 = (K_1 \cup K_0) \setminus K_1$ and $K_{2,2} = (K_1 \cup K_2) \setminus K_1$ have the form $(K_1 \cup K_1) \setminus K_1$. Now we may assume that one of $X$ and $Y$ contains $K_2$, say $X$. Therefore, Lemma 13 implies that $Y$ is $(3K_1, K_2 \cup K_1, K_{1,2})$-free and thus $Y \in (2K_1, K_4 \mid s \geq 1)$ due to Lemma 4. If $Y = K_1$ with $s \geq 2$, then Lemma 13 implies that $X$ is $(3K_1, K_2 \cup K_1, K_{1,2})$-free. Thus, Lemma 4 means that $X = K_1$ with $r \geq 2$ since $X$ contains $K_2$. Therefore, $G = K_{n} = (K_{n-1} \cup K_0) \setminus K_1$ with $n \geq 4$. If $Y = 2K_1$, then Lemma 13 indicates that $X$ is $(3K_1, K_{1,2}, K_3)$-free. Hence, $X \in \{2K_2, K_2 \cup K_1, K_2\}$ due to Lemma 4, and thus $G \in \{2K_2 \setminus K_2, (K_2 \cup K_1) \setminus 2K_1, K_2 \setminus 2K_1 = K_1 \setminus K_1, K_2 \setminus 2K_1\}$.

In what follows, we consider the case of $Y = K_1$, that is $G = X \setminus K_1$. Since $G$ is $K_{1,2}$-free according to Lemma 13, we have $X$ is $(3K_1)$-free and $X$ has at most two connected components. Suppose that $X$ has two connected components, say $X_1$ and $X_2$ with $|X_1|, |X_2| \geq 1$. Then both $X_1$ and $X_2$ are $P_3$-free since otherwise $X$ has an induced $3K_1$, and so $X_1$ and $X_2$ are complete graphs. Therefore, $G = (K_3 \cup K_4) \setminus K_1$ with $s + t = n - 1$ and $s, t \geq 1$. Next we may assume that $X$ is connected. Since $X$ is $P_4$-free, we have $X = X_1 \setminus Y_1$ with $|X_1|, |Y_1| \geq 1$ from Lemma 3. If both $X_1$ and $Y_1$ have no edges, then $X$ is a bipartite graph and so $X \in \{K_2, K_{1,2}, K_{2,2}\}$ by Lemma 12. Note that Lemma 13 means that $G$ is $K_1 \setminus K_{2,2}$-free. Thus, $G \in \{K_3, K_1 \setminus K_1, K_{2,2}\}$. Now we may assume that $X_1$ contains a $K_2$. Then $Y_1$ is $(3K_1, K_2 \cup K_1, K_{1,2})$-free by Lemma 13. Hence,
Y_1 \in \{ K_s, 2K_1 \} by Lemma 4. If Y_1 = K_s with s \geq 2, then X_1 is \{ 3K_1, K_2 \cup K_1, K_{1,2} \}-free by Lemma 13. By Lemma 4, we have X_1 = K_t(t \geq 2) since X_1 has an edge. Note that G = XV_{Y_1} = (X \cup Y_1) \setminus V_{K_1}. Therefore, G = (K_s \setminus V_{K_1}) \cup V_1 \setminus V_{K_1} for n \geq 5. If Y_1 = 2K_1, then G = X_1 \cup V_{2K_1} \cup V_1 = X_1 \cup V_{K_{1,2}}. Lemma 13 indicates that X_1 is \{ 2K_1, K_2 \}-free, and thus X_1 = K_1. Therefore, G = K_1 \cup V_{K_{1,2}}. If Y_1 = K_1, then G = X_1 \cup V_1 \setminus V_{K_1} = X_1 \cup V_K. Therefore X_1 is \{ 3K_1, K_2 \cup K_1, K_{1,2} \}-free by Lemma 13. Lemma 4 indicates that X_1 = K_t(t \geq 2) since X_1 has an edge. Thus, G = K_t \cup V_{K_{n-1}} \cup V_{K_0} \setminus V_{K_1} with n \geq 4.

The proof is completed. \square

In what follows, we detect all mixed graphs whose smallest eigenvalues are greater than \(-\frac{1+\sqrt{5}}{2}\) by considering one by one all the possible underlying graphs.

**Lemma 15.** Let M_C be a mixed graph with G = (K_2 \cup K_1)V_{2K_1}. If any mixed triangle of M_C belongs to C_3 and any induced mixed quadrangle of M_C belongs to C_4, then M_C \in \{ H_1, ..., H_9 \} shown in the Appendix.

**Proof.** Let \(V(G) = \{ v_1, v_2, ..., v_5 \}\) (see the Appendix). Clearly, M_C has two induced quadrangles M_C[\{ v_1, v_2, v_3, v_4 \}] and M_C[\{ v_1, v_5, v_3, v_4 \}]. Note that any induced mixed quadrangle belongs to C_4. We divide four cases to discuss.

**Case 1.** One of them is C_{1,4}^2.

In this case, we may assume M_C[\{ v_1, v_2, v_3, v_4 \}] = C_{1,4}^2. Clearly, there are two different orientations of the mixed cycle M_C[\{ v_1, v_2, v_3, v_4 \}] = \{ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \} and \{ v_1 \rightarrow v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_1 \}. If the former happens, then M_C[\{ v_1, v_2, v_3, v_4 \}] = C_{1,2}^2, and thus \(v_1 \rightarrow v_2, v_3 \rightarrow v_4, v_2 \rightarrow v_1\) since any induced mixed triangle belongs to C_3. It yields that M_C = H_1. If the latter happens, then M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,2}^2 or C_{1,4}^2. Therefore, we can easily verify that M_C = H_2 when M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,2}^2, and M_C = H_3 when M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,4}^2.

**Case 2.** M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,2}^2.

In this case, there are also two different orientations of M_C[\{ v_1, v_2, v_3, v_4 \}] = \{ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \} and \{ v_1 \rightarrow v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_1 \}. Therefore, we can easily verify that M_C = H_4 when the former happens and M_C = H_5 when the latter happens by noticing that any mixed triangle belongs to C_3.

**Case 3.** M_C[\{ v_1, v_2, v_3, v_4 \}] = M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,2}^2.

In this case, there are three different orientations of M_C[\{ v_1, v_2, v_3, v_4 \}] = \{ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \} and \{ v_1 \rightarrow v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_1 \}. Therefore, we can easily verify that M_C = H_6 when the first case happens, M_C = H_7 when the second case happens, and M_C = H_8 when the third case happens.

**Case 4.** M_C[\{ v_1, v_2, v_3, v_4 \}] = C_{1,4}^2 and M_C[\{ v_1, v_5, v_3, v_4 \}] = C_{1,2}^2.

In this case, there are three different orientations of M_C[\{ v_1, v_2, v_3, v_4 \}] = \{ v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow v_4 \} and \{ v_1 \rightarrow v_4 \rightarrow v_3 \rightarrow v_2 \rightarrow v_1 \}. Therefore, we can easily verify that M_C = H_6 when the second case happens, and M_C = H_8 when the third case happens.

This completes the proof. \square

As similar to Lemma 15, we present the following result but omit the tautological proof.

**Lemma 16.** Let M_C be a mixed graph with G = 2K_2 \cup V_{2K_1}. If any mixed triangle of M_C belongs to C_3 and any induced mixed quadrangle belongs to C_4, then M_C \in \{ H_{10}, ..., H_{20} \} shown in the Appendix.

The coalescence \( M \circ_{u,v} M' \) of two mixed graphs M and M' is obtained from M \cup M' by identifying a vertex u of M with a vertex v of M'.

**Lemma 17.** Let G be a connected graph with a cut vertex v such that G - v = G_1 \cup G_2 with V_1 = V(G_1) and V_2 = V(G_2). If \(G_1^+ = G[V_1 \cup \{ v \}]\) and \(G_2^+ = G[V_2 \cup \{ v \}]\), then \([G] = [M \circ_{u,v} M' | M \in [M_{G_1^+}], M' \in [G_2^+]].\)

**Proof.** It is clear that \(G = G_1^+ \circ_{u,v} G_2^+\). For any \(M \circ M'\) with \(M \in [M_{G_1^+}]\) and \(M' \in [G_2^+]\), there exist diagonal matrices D_1 and D_2 with diagonal entries in \([\pm 1, \pm i]\) such that \(D_1H(M)D_1^* = H(G_1^+)\) and \(D_2H(M')D_2^* = H(G_2^+)\). Note that the v-th diagonal entries of D_1 and D_2 satisfy \(D_2(v) = eD_1(v)\) for some \(e \in \{ \pm 1, \pm i \}\). Let D be the diagonal matrix indexed by V(G) such that the diagonal entries are \(D(v_1) = D_1(v_1)\) for \(v_1 \in V_1 \cup \{ v \}\) and \(D(v_2) = eD_2(v_2)\) for \(v_2 \in V_2\). Therefore, one can easily verify that \(DH(M \circ_{u,v} M')D'^* = H(G_1^+ \circ_{u,v} G_2^+) = H(G)\), and thus \(M \circ_{u,v} M' \in [G]\).

Conversely, for any \(M \in [G]\), there exists diagonal matrix D such that \(DH(M)D^* = H(G)\). Note that \(M = M \circ_{u,v} M'\) where \(M = M_{G_1}[V_1 \cup \{ v \}]\) and \(M' = M_{G_2}[V_2 \cup \{ v \}]\). Let D_1 and D_2 be the diagonal matrices indexed by V_1 \cup \{ v \} and V_2 \cup \{ v \} respectively such that the diagonal entries are \(D_1(v_1) = D(v_1)\) for \(v_1 \in V_1 \cup \{ v \}\) and \(D_2(v_2) = D(v_2)\) for \(v_2 \in V_2 \cup \{ v \}\). Therefore, one can easily verify that \(D_1H(M)D_1^* = H(G_1^+)\) and \(D_2H(M')D_2^* = H(G_2^+)\), and thus \(M \in [G_1^+]\) and \(M' \in [G_2^+]\). \square

Now we are ready to present our main result.

**Theorem 5.** Let M_C be a connected mixed graph on n vertices. Then \(\lambda_{\min} > -\frac{1+\sqrt{5}}{2}\) if and only if M_C \in \{ H_1 \cup H_2 \cup H_3 \cup H_4 \}, where
\[
\begin{align*}
H_1 &= \{C_1^2, C_2^2, C_3^2, H_1, H_2, \ldots, H_{27}\}, \\
H_2 &= \{M \cdot u \cdot v \mid u \in V(M), v \in V(M'), M \in [K_3], M' \in [K_3] \cup [K_4]\}, \\
H_3 &= [K_3] = [K_3[s, t] \mid s, t \geq 0, s + t = n], \\
H_4 &= \{M \cdot u \cdot v \mid u \in V(M), v \in V(M'), M \in [K_2], M' \in [K_{n-1}]\}.
\end{align*}
\]

**Proof.** To prove the sufficiency, it only needs to show that each graph in \(H_1 \cup H_2 \cup H_3 \cup H_4\) has smallest eigenvalue greater than \(-1 - \sqrt{2}\). By immediate calculations, the smallest eigenvalues of \(C_1^2, C_2^2, C_3^2\) are all \(-\sqrt{2} > -1 - \sqrt{2}\) and the smallest eigenvalues of \(H_1, H_2, \ldots, H_{27}\) are all \(-1.56 > -1 - \sqrt{2}\) (see the Appendix). For any \(M \in [K_3]\) and \(M' \in [K_3]\), Lemma 17 implies that \(\text{Sp}(M) \cdot M' = \text{Sp}(K_3) \cdot M\). Thus, we have \(\text{Sp}(M) \cdot M' = [2.56, 1, [-1]^2, -1.56]\) by immediate calculations. Similarly, if \(M \in [K_3]\) and \(M' \in [K_4]\), we have \(\text{Sp}(M) \cdot M' = [3.26, 1.34, [-1]^3, -1.60]\). Theorem 2 implies that \(K_{n}[s, t]\) has smallest eigenvalue \(-1\). For any \(M \in H_4\), Lemma 17 implies that \(\text{Sp}(M) = \text{Sp}(K_{n-1} \cdot M), M' \in [K_2] \), whose smallest eigenvalue is the smallest root of \(\varphi(x) = x^3 + (3 - n)x^2 + (1 - n)x - 1 = 0\). Note that \(\varphi(1) = 0, \varphi(-1 - \sqrt{2}) = 1 - n < 0\) for \(n \geq 2\). The smallest root \(\varphi(x)\) is greater than \(-1 - \sqrt{2}\) by the image of \(\varphi(x)\), and thus \(\lambda_{\text{min}}(M) > -1 - \sqrt{2}\).

In what follows, we show the necessity. Since \(\lambda_{\text{min}}(M) > -1 - \sqrt{2}\), Lemmas 7 and 8 indicate that any mixed triangle of \(M\) belongs to \(C_3\) and any mixed induced quadrant of \(M\) belongs to \(C_4\). From Lemma 14, the underlying graph \(G\) belongs to
\[
\{K_{2,2}, K_1 \sqcup K_{1,2}, 2K_2 \sqcup 2K_1, (K_2 \cup K_1) \sqcup 2K_1\} \bigcup \{(K_3 \cup K_1) \sqcup V_1 | s + t = n - 1\}.
\]

If \(G = K_{2,2}\), then \(M \in C_1^2, C_2^2, C_3^2\) \(\subseteq H_1\) due to Lemma 8. If \(G = K_1 \sqcup K_1\), then \(G\) contains no induced cycle with length greater than 3. Thus, Theorem 1 implies that \(M \in [K_1] \cup K_{2,2} = \{H_{21}, \ldots, H_{27}\} \subseteq H_1\). If \(G = 2K_2 \sqcup 2K_1\) or \((K_2 \cup K_1) \sqcup 2K_1\), then \(M \in [H_{21}, \ldots, H_{27}] \subseteq H_1\) due to Lemmas 15 and 16.

If \(G = (K_3 \cup K_1) \sqcup V_1\) with \(s \geq 0\) or \(t = 0\), then \(G = K_3\). Since any mixed triangle of \(M\) belongs to \(C_3\), Theorem 2 means that \(M = M_{K_3} = \{K_3[s, t] \mid s, t \geq 0, s + t = n\} = [K_3] = H_3\). Now we suppose that \(G = K_3 \cup K_1 \sqcup V_1\) with \(s, t \geq 1\). Theorem 1 implies that \(G = [K_3, K_1] \sqcup V_1\) with \(s \geq 1, t \geq 1\). The Hermitian matrix of \(G\) is
\[
H(G) = \begin{pmatrix}
J_S - I_S & 1_S & 0_{S 	imes T} \\
1_S & I_T & 0_{T \times S} \\
0_{S \times T} & 0_{T \times S} & 1_T
\end{pmatrix},
\]
where \(J, I, 1\) and \(0\) are respectively the all-one matrix, identity matrix, all-one vector and zero matrix with the corresponding sizes. Therefore, Lemma 2 indicates that \(\pi\) is an equitable partition with quotient matrix
\[
H_{\pi} = \begin{pmatrix}
s - 1 & 1 & 0 \\
0 & t & 0 \\
0 & 1 & t - 1
\end{pmatrix}.
\]

Assume that \(V_1 = \{v_1, v_2, \ldots, v_s\}\) and \(V_2 = \{u_1, u_2, \ldots, u_t\}\). For \(1 \leq j \leq s\) and \(1 \leq k \leq t\), let \(\delta_{1,j} \in \mathbb{R}^s\) be the vector indexed by \(V_1\) such that \(\delta_{1,j}(v_j) = 1, \delta_{1,j}(v_j') = -1\) and \(\delta_{1,j}(v_j')j \neq 1, j \neq k\) and let \(\delta_{2,k}\) be the vector indexed by \(V_2\) such that \(\delta_{2,k}(u_1) = 1, \delta_{2,k}(u_k) = -1\) and \(\delta_{2,k}(u_k) = 0\) for \(k \neq 1, k\). It is easy to see that \(H(G)\delta_{1,j} = -\delta_{1,j}\) and \(H(G)\delta_{2,k} = -\delta_{2,k}\) for any \(j, k\). Thus, \(H\) has an eigenvalue \(-1\) with multiplicity at least \(s + t - n = n - 3\). Lemma 2 implies that the other three eigenvalues of \(G\) are just the roots \(\lambda_{1} \geq \lambda_{2} \geq \lambda_{3}\) of the function \(f(x) = \det(xI - B_2) = x^3 + (2 - t - s)X^2 + (st - 2t - 2s + 1)X - s - t - 2st\), and thus \(\lambda_{3} = \lambda_{\text{min}}(G) = -\frac{1 + \sqrt{5}}{2}\). It is clear that \(f(0) = s - t - 2s \geq 0\). Note that \(\lambda_{1} > 0\). By the image of the function \(f(x)\), we have \(f\left(-\frac{1 + \sqrt{5}}{2}\right) < 0\). If \(t \geq 3\) then
\[
f\left(-\frac{1 + \sqrt{5}}{2}\right) = \frac{3 - \sqrt{5}}{2} > 0,
\]
a contradiction. Thus, we have \(t \leq 2\). If \(t = 2\) then \(f\left(-\frac{1 + \sqrt{5}}{2}\right) = \frac{3 - \sqrt{5}}{2} > 0\). It leads to \(s < \frac{5 + \sqrt{5}}{3} \approx 3.62\). Thus, we have \(s = 2, 3\) since \(s \geq t\). It means \(M \in [K_2 \cup K_3] \cup (K_3 \cup K_2) \cup K_1\) = \(M \cdot u \cdot v M' \mid M \in [K_3], M' \in [K_3] \cup [K_4] = H_2\).

If \(t = 1\) then \(f\left(-\frac{1 + \sqrt{5}}{2}\right) = -1 < 0\) always holds. Thus, \(s \geq t = 1\) and \(M \in [K_3 \cup K_1] \cup V_1\) = \(M \cdot u \cdot v M' \mid M \in [K_2], M' \in [K_{n-1}] = H_4\).

This completes the proof. □

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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Appendix A. The mixed graphs $C_4$ and $H_1, \ldots, H_{27}$ with their smallest eigenvalues